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Some Summation Formulas for the Series ${}_3F_2(1)$

By J. L. Lavoie

Abstract. Summation formulas, contiguous to Watson's and Whipple's theorems in the theory of the generalized hypergeometric series, are obtained. Certain limiting cases of these results are given.

1. Two Results Contiguous to Watson's Theorem. The two following summation formulas for series ${}_3F_2(1)$ are useful, interesting, easily established, and probably new. They are

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c-1 \end{matrix}\middle| 1\right) \\
&\quad = \frac{2^{a+b}\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)} \\
&\quad\times\left(\frac{\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}b+1)}{\Gamma(c-\frac{1}{2}a-\frac{1}{2})\Gamma(c-\frac{1}{2}b-\frac{1}{2})}+\frac{ab\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{4\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b)}\right), \\
&\qquad\qquad R(2c-a-b)>1,
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \end{matrix}\middle| 1\right) \\
&\quad = \frac{2^{a+b}\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(a+1)\Gamma(b+1)} \\
&\quad\times\left(\frac{\Gamma(\frac{1}{2}a+1)\Gamma(\frac{1}{2}b+1)}{\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})}-\frac{ab\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}{4\Gamma(c-\frac{1}{2}a+1)\Gamma(c-\frac{1}{2}b+1)}\right), \\
&\qquad\qquad R(2c-a-b)>-3.
\end{aligned}
\tag{2}
$$

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They are thus seen to be contiguous to Watson's theorem [1, p. 16, 3.3.1],

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\middle|1\right)
= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})},
\quad R(2c-a-b) > -1.
$$

Proofs. It is just a simple exercise to show that the left-hand side reduces to the right-hand side in the following relation involving three ${}_3F_2$:

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c-1 \end{matrix}\middle|1\right) - {}_3F_2\left(\begin{matrix} a, & b, & c-1 \\ \frac{1}{2}(a+b+1), & 2c-2 \end{matrix}\middle|1\right)
= \frac{ab}{(a+b+1)(2c-1)}\,{}_3F_2\left(\begin{matrix} a+1, & b+1, & c \\ \frac{1}{2}(a+b+3), & 2c \end{matrix}\middle|1\right).
$$

But two of these ${}_3F_2$ can be evaluated by Watson's theorem, and (1) is obtained when we make use of various familiar identities relating to the Γ-functions.

In exactly the same way, (2) is obtained from the relation

$$
{}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 \end{matrix}\middle|1\right) - {}_3F_2\left(\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}\middle|1\right)
= -\frac{ab}{(a+b+1)(2c+1)}\,{}_3F_2\left(\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3), & 2c+2 \end{matrix}\middle|1\right).
$$

2. Two Results Closely Related to Whipple's Theorem. Formulas (1) and (2) lead, respectively, to the two summation formulas:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\middle|1\right) \\
(3)\qquad &= \frac{\Gamma(e)\Gamma(f)}{2^{2a+1}\Gamma(e-a)\Gamma(f-a)} \\
&\times\left(\frac{\Gamma[\frac{1}{2}(e-a)]\Gamma[\frac{1}{2}(f-a)]}{\Gamma[\frac{1}{2}(e-b)]\Gamma[\frac{1}{2}(f-b)]} + \frac{\Gamma[\frac{1}{2}(e-a+1)]\Gamma[\frac{1}{2}(f-a+1)]}{\Gamma[\frac{1}{2}(e-b+1)]\Gamma[\frac{1}{2}(f-b+1)]}\right),
\end{aligned}
$$

provided the parameters satisfy the conditions $a+b=0$ and $e+f=1+2c$ with $R(c) > -1$, and

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\middle|1\right) \\
(4)\qquad &= \frac{\Gamma(e)\Gamma(f)}{2^{2a-1}(a-1)(c-1)\Gamma(e-a)\Gamma(f-a)} \\
&\times\left(\frac{\Gamma[\frac{1}{2}(e-a)]\Gamma[\frac{1}{2}(f-a)]}{\Gamma[\frac{1}{2}(e-b)]\Gamma[\frac{1}{2}(f-b)]} - \frac{\Gamma[\frac{1}{2}(e-a+1)]\Gamma[\frac{1}{2}(f-a+1)]}{\Gamma[\frac{1}{2}(e-b+1)]\Gamma[\frac{1}{2}(f-b+1)]}\right),
\end{aligned}
$$

provided $a+b=2$ and $e+f=1+2c$ with $R(c) > 1$.

These results are thus closely related to Whipple's theorem [1, p. 16, 3.4.1]:

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\middle|1\right) \\
&\quad = \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma\left[\frac{1}{2}(e+a)\right]\Gamma\left[\frac{1}{2}(f+a)\right]\Gamma\left[\frac{1}{2}(e+b)\right]\Gamma\left[\frac{1}{2}(f+b)\right]},
\end{aligned}
$$

where $a + b = 1$ and $e + f = 1 + 2c$ with $R(c) > 0$.

Proofs. Consider the following familiar transformation [1, p. 14]:

$$
(5)\quad {}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\middle|1\right) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\,{}_3F_2\left(\begin{matrix} e-a, & f-a, & s \\ & s+b, & s+c \end{matrix}\middle|1\right),
$$

where $s = e + f - a - b - c$. As is shown on page 16 of [1], Watson's theorem can be used to sum the series on the right of (5), provided that $a + b = 1$ and $e + f = 1 + 2c$, and Whipple's theorem is obtained. Similarly, when $a + b = 0$ and $e + f = 1 + 2c$, and when $a + b = 2$ and $e + f = 1 + 2c$, (1) and (2) can be used to evaluate the ${}_3F_2(1)$ on the right of (5), thus yielding (3) and (4), respectively.

3. Some Limiting Cases. If the parameters are such that the ${}_3F_2(1)$ in

$$
(6)\quad {}_4F_3\left(\begin{matrix} 1+a, & 1+b, & 1+c, & 1 \\ & 1+e, & 1+f, & 2 \end{matrix}\middle|1\right) = \frac{ef}{abc}\left({}_3F_2\left(\begin{matrix} a, & b, & c \\ & e, & f \end{matrix}\middle|1\right) - 1\right)
$$

can be summed by (2), then letting $c \to 0$ and using L'Hospital's rule yields, with $R(a + b) < 3$,

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} 1+a, & 1+b, & 1, & 1 \\ \frac{1}{2}(a+b+3), & 2, & 2 \end{matrix}\middle|1\right) \\
(7)\quad &= \frac{1+a+b}{2ab}\left\{\psi(1) + \psi\left(\tfrac{1}{2} - \tfrac{1}{2}a - \tfrac{1}{2}b\right) - \psi(1-a) - \psi(1-b)\right. \\
&\qquad + \frac{\cos\pi(a-b)/2}{2\cos\pi(a+b)/2}\left(\psi\left(1 - \tfrac{1}{2}a\right) + \psi\left(1 - \tfrac{1}{2}b\right)\right. \\
&\qquad\qquad \left.\left. -\psi\left(\tfrac{1}{2} - \tfrac{1}{2}a\right) - \psi\left(\tfrac{1}{2} - \tfrac{1}{2}b\right)\right)\right\}.
\end{aligned}
$$

If the parameters are such that the ${}_3F_2(1)$ in (6) can be evaluated by (3), then a formula resembling (7) is obtained when $c \to 0$. The result is given in [4, (2)].

The three following special cases are similarly derived when the ${}_3F_2$ in (6) is summed by (1), or by (2), or by Watson's theorem. Letting $b \to 0$, we get

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} 1+a, & 1+c, & 1, & 1 \\ \frac{1}{2}(a+3), & 2c, & & 2 \end{matrix}\middle|1\right) \\
&= \frac{(a+1)(2c-1)}{4ac}\left\{\psi\left(\tfrac{1}{2}a + \tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2}\right) + \psi\left(c - \tfrac{1}{2}\right) - \psi\left(c - \tfrac{1}{2}a - \tfrac{1}{2}\right)\right\} \\
&\quad + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{3}{2}\right)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}a + 1\right)\Gamma\left(c - \frac{1}{2}a\right)\Gamma(c+1)}, \qquad R(2c - a) > 1,
\end{aligned}
$$

or

$$
{}_4F_3\left(\begin{matrix} 1+a, & 1+c, & 1, & 1 \\ \frac{1}{2}(a+3), & 2+2c, & 2 \end{matrix}\middle| 1\right)
$$
$$
= \frac{(a+1)(2c+1)}{4ac}\left\{\psi\left(\tfrac{1}{2}a+\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2}\right) + \psi\left(c+\tfrac{1}{2}\right) - \psi\left(c-\tfrac{1}{2}a+\tfrac{1}{2}\right)\right\}
$$
$$
- \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{3}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c+\frac{3}{2}\right)}{2c\Gamma\left(\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma(c+1)}, \qquad R(2c-a) > -3,
$$

or

$$
{}_4F_3\left(\begin{matrix} 1+a, & 1+c, & 1, & 1 \\ \frac{1}{2}(a+3), & 1+2c, & 2 \end{matrix}\middle| 1\right)
$$
$$
= \frac{a+1}{2a}\left\{\psi\left(\tfrac{1}{2}a+\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2}\right) + \psi\left(c+\tfrac{1}{2}\right) - \psi\left(c-\tfrac{1}{2}a+\tfrac{1}{2}\right)\right\},
$$
$$
R(2c-a) > -1.
$$

A further result of a similar nature can be obtained by using (4) on the right of (6). After letting $a \to 0$, we find that

$$
{}_4F_3\left(\begin{matrix} 1+c, & 3, & 1, & 1 \\ 1+e, & 1+f, & 2 \end{matrix}\middle| 1\right)
$$
$$
= \frac{ef}{2(c-1)}\left\{1 - \frac{(e-1)(f-1)-(c-1)}{2c}\right.
$$
$$
\left.\times\left[\psi\left(\tfrac{1}{2}e+\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2}e\right) + \psi\left(\tfrac{1}{2}f+\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2}f\right)\right]\right\},
$$

with $f = 1 + 2c - e$, $R(c) > 1$.

If (3) is used on the right of (6) and $a \to 0$, then [4, (3)] is obtained. Similarly, letting $a \to 1$ in (4), or using Whipple's theorem in (6) and letting $a \to 0$, yields [4, (1)]. This last formula is, essentially, a result given by Watson in 1917, [6, p. 245] or [1, p. 98, Example 9].

Incidentally, consider the formula

$$
{}_3F_2\left(\begin{matrix} 1, & 1+a, & a+b \\ & 1+b, & 1+a+b \end{matrix}\middle| 1\right)
$$
$$
= \frac{b(b+a)}{a}\left\{\psi(b+a) - \psi(b-a) - \psi\left(\frac{1+b+a}{2}\right) + \psi\left(\frac{1+b-a}{2}\right)\right\}, \tag{8}
$$
$$
R(b-a) > 0,
$$

which appears to have been first explicitly stated by Hardy [2, (8.4)], where the series is not written in hypergeometric form. After a change of variables, this can be transformed, by (5), into the following companion formula to Watson's result:

$$
{}_3F_2\left(\begin{matrix} 1+\nu-\mu, & \nu+\mu, & \nu+\frac{1}{2} \\ & 2\nu+1, & \nu+\frac{3}{2} \end{matrix}\middle| 1\right)
$$
$$
= \frac{\Gamma(2\nu+2)}{(2\mu-1)\Gamma(\nu+\mu)\Gamma(1+\nu-\mu)} \tag{9}
$$
$$
\times\left\{\psi(\nu+\mu) - \psi(1+\nu-\mu) - \psi\left(\frac{1}{2}+\frac{\nu+\mu}{2}\right) + \psi\left(1+\frac{\nu-\mu}{2}\right)\right\}.
$$

Although obtained by a different technique, this is essentially the last formula given in [4].

Finally, letting $c \to \infty$ in (3) and (4) yields

$$(10)\quad {}_2F_1\left(\begin{matrix} a, & -a \\ & e \end{matrix} \middle| \tfrac{1}{2}\right) = \frac{\Gamma(e)}{2^{a+1}\Gamma(e-a)} \left\{ \frac{\Gamma[\frac{1}{2}(e-a)]}{\Gamma[\frac{1}{2}(e+a)]} + \frac{\Gamma[\frac{1}{2}(e-a+1)]}{\Gamma[\frac{1}{2}(e+a+1)]} \right\}$$

and

$$(11)\quad \begin{aligned} &{}_2F_1\left(\begin{matrix} a, & 2-a \\ & e \end{matrix} \middle| \tfrac{1}{2}\right) \\ &= \frac{\Gamma(e)}{2^{a-1}(a-1)\Gamma(e-a)} \left\{ \frac{\Gamma[\frac{1}{2}(e-a)]}{\Gamma[\frac{1}{2}(e+a-2)]} - \frac{\Gamma[\frac{1}{2}(e-a+1)]}{\Gamma[\frac{1}{2}(e+a-1)]} \right\}. \end{aligned}$$

These are closely related to Kummer's formula [3, p. 134]:

$${}_2F_1\left(\begin{matrix} a, & 1-a \\ & e \end{matrix} \middle| \tfrac{1}{2}\right) = \frac{\Gamma(\frac{1}{2}e)\Gamma(\frac{1}{2}+\frac{1}{2}e)}{\Gamma(\frac{1}{2}e+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}a)},$$

obtained by letting $c \to \infty$ in Whipple's theorem. A very simple way of obtaining these special cases directly is to use the contiguous function relations for the ${}_2F_1$. They are given explicitly by Rainville [5, p. 71, Exercises 21], and it is easily verified that

$$F\left(\begin{matrix} a, & -a \\ & e \end{matrix} \middle| \tfrac{1}{2}\right) = \tfrac{1}{2}F\left(\begin{matrix} a, 1-a \\ e \end{matrix} \middle| \tfrac{1}{2}\right) + \tfrac{1}{2}F\left(\begin{matrix} a+1, -a \\ e \end{matrix} \middle| \tfrac{1}{2}\right)$$

and

$$F\left(\begin{matrix} a, & 2-a \\ & e \end{matrix} \middle| \tfrac{1}{2}\right) = \frac{a+e-2}{a-1}F\left(\begin{matrix} a; 1-a \\ e \end{matrix} \middle| \tfrac{1}{2}\right) - \frac{a+e-1}{a-1}F\left(\begin{matrix} a+1, -a \\ e \end{matrix} \middle| \tfrac{1}{2}\right).$$

The formulas (10) and (11) are obtained when the series on the right are summed by Kummer's theorem.

I am indebted to the referee for pointing out that from Legendre's complete elliptic integrals

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2} \\ & 1 \end{matrix} \middle| k^2\right), \quad E(k) = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix} -\frac{1}{2}, & \frac{1}{2} \\ & 1 \end{matrix} \middle| k^2\right), \qquad |k^2| < 1,$$

we obtain, from (10) and (11), that

$$E(2^{-1/2}) = \frac{[\Gamma(\frac{1}{4})]^2}{8\pi^{1/2}} + \frac{\pi^{3/2}}{[\Gamma(\frac{1}{4})]^2} = \frac{1}{2}K(2^{-1/2}) + \frac{\pi}{4K(2^{-1/2})}.$$

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Département de Mathématiques, Statistique et Actuariat
Université Laval, Cité Universitaire
Québec, Canada G1K 7P4

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